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SAN DIEGO

REPORT ZP-7-022 TN

DATE 2-14-56

MODEL XM-65

TITLE

ANALYSIS OF MISSILE TANK GEOMETRIES

ZP-7-022 TN

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PREPARED BY F. D'Vincent

GROUP Model 7 - Preliminary Design and Systems Analysis

REFERENCE

CHECKED BY

APPROVED BY Krafft A. Ehricke

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FOREWORD

This is a technical note pertaining to missile tank geometry. It provides a list of formulas that may be referred to in reports and other technical papers where their development is incidental.

TABLE OF CONTENTS

SUMMARY

Mathematical relations are presented pertaining to various missile tank and bulkhead configurations. Although some of these relations are available in handbooks, they are included in this technical note together with a number of newly derived formulas. In this manner a comprehensive summary of equations is obtained, permitting rapid computation of volumes, surfaces and other geometric characteristics of a great variety of shapes.

The purpose of this technical note is to eliminate the need for repeating the derivations of frequently used equations in preliminary design and layout of missiles.

INTRODUCTION

Reports concerning missile tank geometry have been compiled and it is expected that studies of this sort will continue. The studies indicate the necessity for a list of reference formulas applying to tank geometry that may be drawn from and referred to, eliminating the need of developing them within a report or the repetition of their development in related reports. Although it is felt proper to make calculations within a report, geometric formula development is incidental at the time of writing and should be available elsewhere.

This technical note is presented for the purpose of fulfilling this requirement and effort has been made to include a broad range of mathematical expressions to meet the needs of missile tank designers.

These formulas presented consist of mathematical expressions for tank and bulkhead volumes, their surface areas and other pertinent geometric information that is not readily available in standard handbooks. Formulas for partially filled tank shapes are included along with those of partially filled bulkheads.

In general, this report is compiled for use with studies pertaining to the SM-65 vehicle, however, it includes geometries for other shape tanks and bulkheads that may be used in future development studies.

No attempt is made by this report to qualify the choice of any particular tank or bulkhead.

INTRODUCTION (cont'd.)

The first part following this introduction contains a comprehensive list of formulas together with reference numbers which may be referred to in technical studies. Each of these formulas refer to its origination whether "classic" or whether derived within this report. If it is derived within this report, its location is referred to by page number.

The second part is devoted to the derivation of the uncommon formulas referred to in the first section. They are presented in a form that gives the designer an understanding of the fundamental derivation and allows him to change the formula's original parameters without time consuming effort.

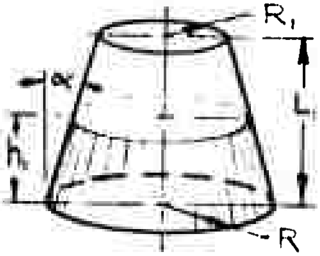
The formulas contained in this report have been checked. However, errors may exist and their appearance should be brought to the authors attention for correction.

PART I

SUMMARY OF EQUATIONS

V_T IS TOTAL VOLUME
S_T IS TOTAL SURFACE AREA

CONE FRUSTUM TANK REF. PAGES 6 & 7

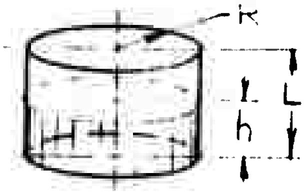


(1) $$V_T = \frac{\pi}{3\tan\alpha}\left(R^3 - R_1^3\right)$$

(2) $$S_T = \frac{\pi L_1 (R_1 + R)}{\cos\alpha} \quad *$$
$$= \pi(R+R_1)\sqrt{L_1^2 + (R-R_1)^2}$$

(3) $$V_{(\text{OF } h_1)} = \pi h_1\left(R^2 - R h_1 \tan\alpha + \tfrac{1}{3} h_1^2 \tan^2\alpha\right)$$

CYLINDRICAL TANK (CLASSIC FORMULAS)

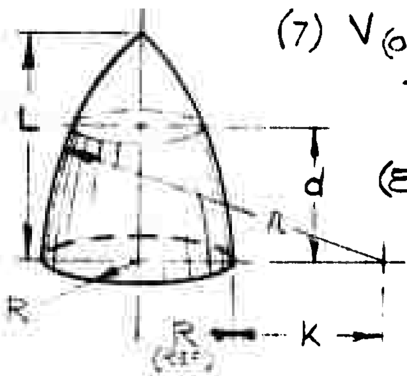


(4) $$V_T = \pi R^2 L$$

(5) $$V_{(\text{OF } h)} = \pi R^2 h$$

(6) $$S_T = 2\pi R L \quad *$$

OGIVE TANK (CIRCULAR) REF. PAGE 8

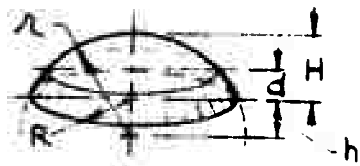


(7) $$V_{(\text{OF } d)} = \pi\left[K^2 d - K\left\{d\sqrt{\lambda^2 - d^2} + \lambda^2 \sin^{-1}\left(\frac{d}{\lambda}\right)\right\} + \lambda^2 d - \tfrac{1}{3} d^3\right]$$

(8) $$S_{(\text{OF } d)} = 2\pi\lambda\left[d - K\sin^{-1}\left(\frac{d}{\lambda}\right)\right] \quad *$$

* NOTE. - TANK ENDS ARE NOT INCLUDED IN SURF. AREAS

SPHERICAL BULKHEAD   REF. PAGES 9, 10 & 11



(9) $V_T = \frac{1}{3}\pi H^2(3r - H)$

(10) $= \frac{1}{6}\pi H(H^2 + 3R^2)$

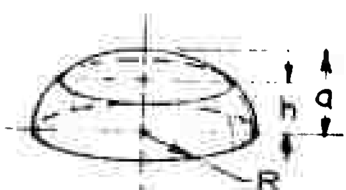
(11) $r = \frac{H}{2} + \frac{R^2}{2H}$

(12) $S_T = 2\pi r H$ *

Use equ 9 TWICE $\cancel{(13)\ V_{(OF\ h)} = \pi d\left[(r^2 - h^2) - d\left(\frac{1}{3}d + h\right)\right]}$
(13) in ERROR

$h = r - H$ OR $h = \sqrt{r^2 - R^2}$

ELLIPTICAL BULKHEAD (IN CYLINDRICAL TANK)   REF. PAGES 12, 13 & 14



(14) $V_T = \frac{2}{3}\pi R^2 a$

(15) $V_{(OF\ h)} = \pi h\left(\frac{R}{a}\right)^2\left(a^2 - \frac{1}{3}h^2\right)$

(16) $S_T = \pi\left\{a\sqrt{C_1 a^2 + R^2} + \frac{R^2}{\sqrt{C_1}}\left[\mathrm{LOG}\left(a\sqrt{C_1} + \sqrt{C_1 a^2 + R^2}\right) - \mathrm{LOG}\,R\right]\right\}$ *

$C_1 = K_1^4 - K_1^2$ WHERE $K_1 = R/a$
LOG IS OF BASE "e"

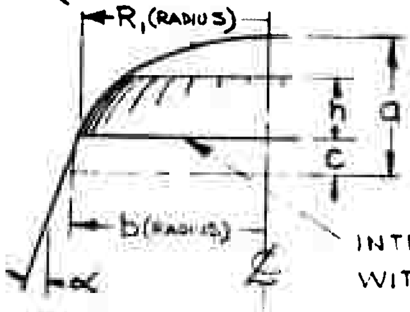
(17) RADIUS OF CURVATURE (ρ) DIST. X ABOVE BASE PLANE

$$\rho = \frac{\left[R^2 + K_1^2 X^2 (K_1^2 - 1)\right]^{3/2}}{K_1^2 R^2}$$

* NOTE :- BULK'D BASE IS NOT INCLUDED IN SURF. AREA

ELLIPTICAL BULKHEAD (IN CONE FRUSTUM TANK)

REF. PAGES 15, 16, 16a, 16b & 17



$$C = \frac{R_1 \tan\alpha}{K_2^2}, \quad K_2 = \frac{b}{a}$$

$$= \frac{a^2 \tan\alpha}{\sqrt{b^2 + a^2 \tan^2\alpha}}$$

$$R_1 = \frac{b}{a}\sqrt{a^2 - c^2}$$

INTERSECTION LINE OF BULKHEAD WITH TANK

α = ANGLE OF CONE FRUSTUM TANK

(18) V_T (TOTAL VOL. OF BULK'D FROM BASE [TANK'S INTERSECTION] TO APEX)

$$= \pi\left(\frac{b}{a}\right)^2\left[a^2(a-c) + \frac{1}{3}(c^3 - a^3)\right]$$

(19) V (VOLUME FROM BASE TO h")

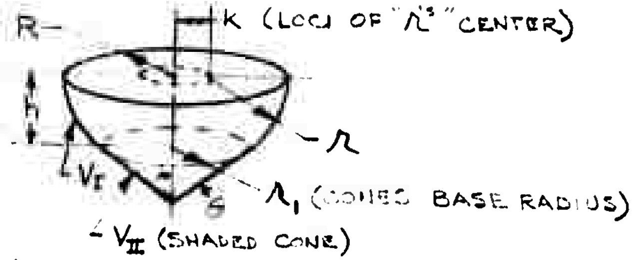
$$= \pi\left(\frac{b}{a}\right)^2\left[a^2(h-c) + \frac{1}{3}(c^3 - h^3)\right]$$

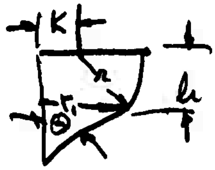
(20) S (SURFACE AREA OF BULK'D FROM BASE [TANK'S INTERSECTION] TO APEX)

$$S = \pi\left[a\sqrt{C_1 a^2 + b^2} + \frac{b^2}{\sqrt{C_2}}\operatorname{LOG}\left(a\sqrt{C_2} + \sqrt{C_2 a^2 + b^2}\right) - c\sqrt{C_1 c^2 + b^2} - \frac{b^2}{\sqrt{C_2}}\operatorname{LOG}\left(c\sqrt{C_2} + \sqrt{C_2 c^2 + b^2}\right)\right]$$

a = BULK'D MINOR SEMI AXIS (HEIGHT)
b = " MAJOR " "
c = DIST BETWEEN MAJOR AXIS AND BULK'D INTERSECTION WITH TANK $= R_1 \tan\alpha / K_2^2$
$C_1 = K_2^4 - K_2^2$ WHERE $K_2 = b/a$
LOG IS BASE "e"

TORICONICAL BULKHEAD   REF. PAGES 18 & 19





(21) $V_T = V_I + V_{II}$

$$V_I = \pi\left\{h(r^2+k^2)+k\left[h\sqrt{r^2-h^2} + r^2\sin^{-1}\left(\frac{h}{r}\right)\right]-\frac{h^3}{3}\right\}$$

$$V_{II} = \tfrac{1}{3}\pi r_1^3 \cot\theta$$

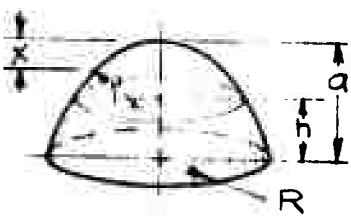
$\left(\sin^{-1}\frac{h}{r} = \theta \text{ RAD.}\right)$

(22) $S = S_I + S_{II}$

$$S_I = 2\pi r\left[K\sin^{-1}\left(\frac{h}{r}\right)+h\right] \quad \text{(TORUS)}$$

$$S_{II} = \frac{\pi r_1^2}{\sin\theta} \quad \text{(CONE)}$$

PARABOLIC BULKHEAD   REF. PAGES 20 & 21



(23) $V_T = \frac{1}{2}\pi R^2 a$

(24) $V_{(\text{of } h)} = \frac{1}{2}\pi R^2 h\left[2-\frac{h}{a}\right]$

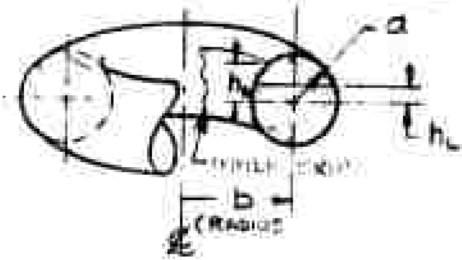
(25) $S_T = \frac{1}{6}\pi R a\left[\left(4+\frac{R^2}{a^2}\right)^{3/2}-\left(\frac{R}{a}\right)^3\right]$

(27) RADIUS OF CURVATURE (ρ)

$$\rho = \frac{1}{2Ra}\left(4ax+R^2\right)^{3/2}$$

X = VERTICAL DIST FROM VERTEX TO WHERE ρ IS DESIRED (ON THE SURFACE)

TORUS TANK   REF PAGES 22 THRU 26



(28) $V_T = 2\pi^2 b a^2 = 19.739\, b a^2$

(29) PARTIALLY FILLED TORUS TO h_L (NOTE:- h_L MAY BE + OR −) VOL. OF LIQUID =

$$V_L = 2\pi a^2 b \left\{ K\sqrt{1-K^2} + \sin^{-1} K + \frac{\pi}{2} \right\}$$

$$K = h_L / a$$

(30) $S_T = 4\pi^2 a b$

(31) OUTER SKIN SURFACE AREA OF $2h_s$ TOTAL WIDTH AND SYMMETRICAL ABOUT THE HORIZONTAL CENTERLINE.

$$S_o = 4\pi a \left[b \sin^{-1}\left(\frac{h_s}{a}\right) + h_s \right]$$

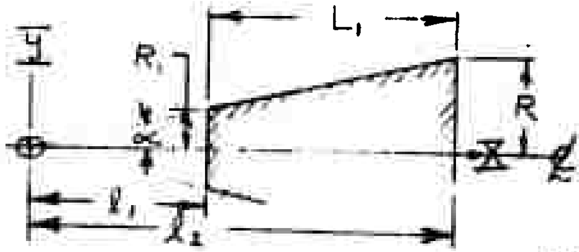
(32) INNER SKIN ($2h_s$ WIDE)

$$S_i = 4\pi a \left[b \sin^{-1}\left(\frac{h_s}{a}\right) - h_s \right]$$

PART II

DERIVATIONS

CONE FRUSTUM TANK



$$L_1 = \ell_2 - \ell_1 = \frac{R - R_1}{\tan\alpha}$$

$$y = x\tan\alpha \qquad \ell_1 = \frac{R_1}{\tan\alpha}, \quad \ell_2 = \frac{R}{\tan\alpha}$$

VOLUME (ENCLOSED FROM R_1 TO R)

$$V_T = \pi\int_{\ell_1}^{\ell_2} y^2\,dx = \pi\int_{\frac{R_1}{\tan\alpha}}^{\frac{R}{\tan\alpha}} x^2\tan^2\alpha\,dx$$

$$= \pi\tan^2\alpha\,\frac{x^3}{3}\Bigg]_{\frac{R_1}{\tan\alpha}}^{\frac{R}{\tan\alpha}}$$

$$= \frac{1}{3}\pi\tan^2\alpha\left(\frac{R^3}{\tan^3\alpha} - \frac{R_1^3}{\tan^3\alpha}\right)$$

$$= \frac{\pi}{3\tan\alpha}\left(R^3 - R_1^3\right) \qquad \text{OR} = \frac{\pi L_1}{3}\left(R^2 + RR_1 + R_1^2\right)$$

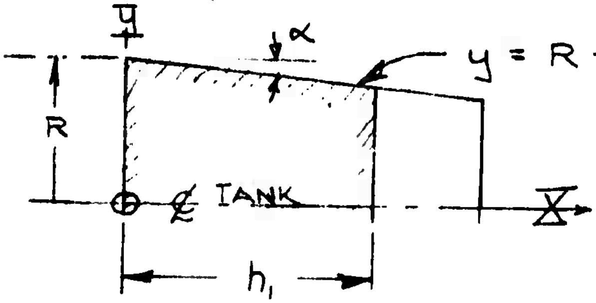
SURFACE AREA

$$S = \left(\frac{R_1 + R}{2}\right)\cdot 2\pi\cdot\frac{L_1}{\cos\alpha}$$

$$= \frac{\pi L_1 (R_1 + R)}{\cos\alpha} \qquad \text{OR} = \pi(R + R_1)\sqrt{L_1^2 + (R - R_1)^2}$$

CONE FRUSTUM TANK (CONT'D)

PARTIALLY FILLED TANK IN TERMS OF BASE RADIUS (R), CONE ANGLE (α) AND HEIGHT OF LIQUID (h) ABOVE TANK'S BASE.



VOLUME 0 TO h_1

$$V = \pi \int_0^h y^2 \, dx$$

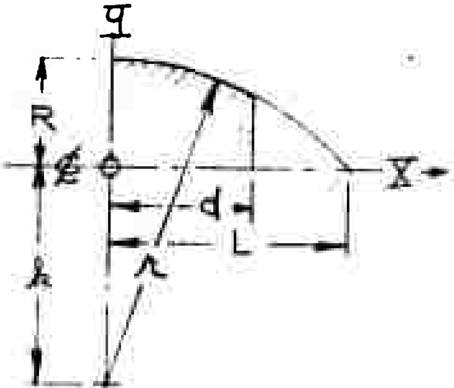
$$= \pi \int_0^h (R - X\tan\alpha)^2 \, dx$$

$$= \pi \int_0^h (R^2 - 2RX\tan\alpha + X^2\tan^2\alpha) \, dX$$

$$= \pi \left[R^2 X - RX^2 \tan\alpha + \tfrac{1}{3} X^3 \tan^2\alpha \right]_0^{h_1}$$

$$= \underline{\underline{\pi h_1 \left[R^2 - Rh_1 \tan\alpha + \tfrac{1}{3} h_1^2 \tan^2\alpha \right]}}$$

OGIVE TANK (CIRCULAR) (RADIUS NOSE)



$$x^2 + (y+k)^2 = r^2$$

$$y = -k + \sqrt{r^2 - x^2}$$

$$\frac{dy}{dx} = -x\,(r^2 - x^2)^{-\frac{1}{2}}$$

VOLUME: (ENCLOSED TO "d")

$$V = \pi \int_0^d y^2\,dx = \pi \int_0^d \left(-k + \sqrt{r^2 - x^2}\right)^2 dx$$

$$= \pi \int_0^d \left(k^2 - 2k\sqrt{r^2 - x^2} + r^2 - x^2\right) dx$$

$$= \pi \Big[k^2 x - 2k \cdot \tfrac{1}{2}\left\{ x\sqrt{r^2 - x^2} + r^2 \sin^{-1}\left(\tfrac{x}{r}\right)\right\} + r^2 x - \tfrac{1}{3}x^3 \Big]_0^d$$

$$= \pi \Big[k^2 d - k\left\{ d\sqrt{r^2 - d^2} + r^2 \sin^{-1}\left(\tfrac{d}{r}\right)\right\} + r^2 d - \tfrac{1}{3} d^3 \Big]$$

SURFACE AREA (TO d)

$$S = 2\pi \int_0^d \left(-k + \sqrt{r^2 - x^2}\right)\left[1 + \frac{x^2}{r^2 - x^2}\right]^{1/2} dx$$

$$= 2\pi \int_0^d \left(-k + \sqrt{r^2 - x^2}\right)\left[\frac{r}{\sqrt{r^2 - x^2}}\right] dx$$

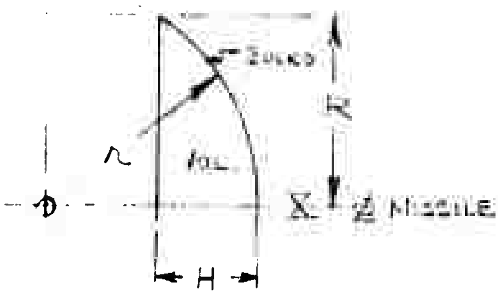
$$= 2\pi \int_0^d \left(\frac{-kr}{\sqrt{r^2 - x^2}} + r\right) dx = 2\pi r\left[-k \sin^{-1}\left(\tfrac{x}{r}\right) + x\right]_0^d$$

$$= 2\pi r \left[d - k \sin^{-1}\left(\tfrac{d}{r}\right)\right]$$

SPHERICAL BULKHEAD

(THE FOLLOWING APPLIES TO A BULK'D FOR A CYLINDRICAL OR TAPERED TANK)

I



R = MISSILE TANK RADIUS
r = BULK'D RADIUS
H = BULK'D HEIGHT

H & R KNOWN
$r = ?$

$$x^2 + y^2 = r^2 \qquad y^2 = r^2 - x^2$$

$$(r - H)^2 + R^2 = r^2 \quad \text{OR} \quad r^2 - 2rH + H^2 + R^2 = r^2$$

$$r = \frac{H}{2} + \frac{R^2}{2H}$$

VOLUME

IN TERMS OF BULK'D HEIGHT (H) & RADIUS (r)

$$V = \pi \int_{r-H}^{r} y^2\,dx = \pi \int_{r-H}^{r} (r^2 - x^2)\,dx$$

$$= \pi \left(r^2 x - \frac{x^3}{3} \right) \Big]_{r-H}^{r}$$

$$= \pi \left[\left(r^3 - \frac{r^3}{3} \right) - \left(r^2 (r-H) - \frac{(r-H)^3}{3} \right) \right]$$

$$= \pi \left[H r^2 - \tfrac{1}{3} r^3 + \frac{(r-H)^3}{3} \right] = \underline{\underline{\frac{H^2}{3} \pi (3r - H)}}$$

IN TERMS OF BULK'D HEIGHT (H) & TANK RAD (R)

$r = \frac{H}{2} + \frac{R^2}{2H}$ IN THE ABOVE FORMULA

$$V = \frac{1}{6} \pi H \left[H^2 + 3R^2 \right]$$

SPHERICAL BULKHEAD (CONT'D)

SURFACE AREA

$$S = 2\pi \int_{r-H}^{r} y \, ds \qquad ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$

$$x^2 + y^2 = r^2$$

$$2x\,dx + 2y\,dy = 0$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

$$S = 2\pi \int_{r-H}^{r} y \sqrt{1 + \left(-\frac{x}{y}\right)^2} \, dx$$

$$= 2\pi \int_{r-H}^{r} \sqrt{y^2 + x^2} \, dx$$

$$= 2\pi \int_{r-H}^{r} \sqrt{r^2 - \cancel{x^2} + \cancel{x^2}} \, dx$$

$$= 2\pi \int_{r-H}^{r} r \, dx$$

$$= 2\pi r x \Big]_{r-H}^{r}$$

$$= 2\pi r \left[r - (r - H) \right]$$

$$= \underline{\underline{2\pi r H}}$$

FOR HEMISPHERE $H = r$

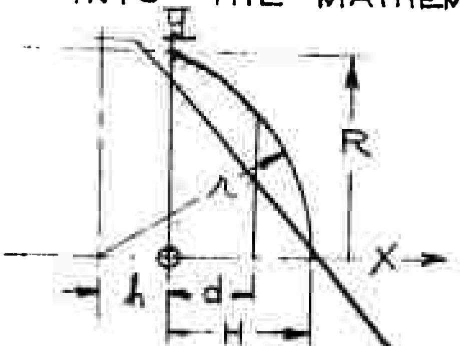
$$S = 2\pi r^2 \quad \text{(HEMISPHERE)}$$

SPHERICAL BULKHEAD (CONT'D)

VOLUME OF LIQUID IN PARTIALLY FILLED BULK'D

IN TERMS OF LIQUID DEPTH (d), BULK'D RADIUS (r) AND DIST. (h) WHICH IS THE DISTANCE BETWEEN THE BULK'D BASE AND THE CENTER POINT OF THE BULK'D RADIUS.

NOTE: IT IS DESIRABLE TO SHIFT THE X AXIS TO THE BULK'DS BASE FOR REASONS OF SIMPLIFICATION WHEN (h) IS INTRODUCED INTO THE MATHEMATICAL EXPRESSION.



$$(x+h)^2 + y^2 = r^2$$

$$y^2 = r^2 - x^2 - 2hx - h^2$$

$$V = \pi \int_0^d y^2\,dx = \pi \int_0^d (r^2 - x^2 - 2hx - h^2)\,dx$$

$$= \pi \left[r^2 x - \tfrac{1}{3}x^3 - hx^2 - h^2 x \right]_0^d$$

$$= \pi d \left(r^2 - \tfrac{1}{3}d^2 - hd - h^2 \right]$$

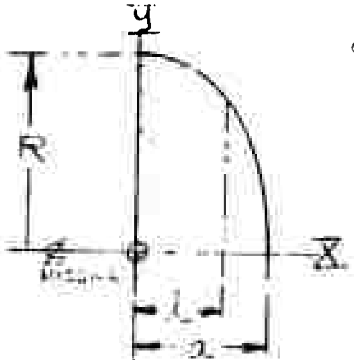
$$= \pi d \left[(r^2 - h^2) - d\left(\tfrac{1}{3}d + h\right) \right]$$

$$h = r - H \quad \text{OR} \quad \sqrt{r^2 - R^2}$$

does not work

ELLIPTICAL BULKHEAD

VOLUME (ENCLOSED BY THE BULKHEAD IN A CYLINDRICAL TANK)



GEN. ELLIP. EQUATION

$$\frac{X^2}{a^2} + \frac{y^2}{R^2} = 1$$

$$y^2 = R^2 - (K_1 X)^2$$

$$K_1 = R/a$$

R = TANK RADIUS
a = BULK'D HEIGHT

VOLUME ENCLOSED TO h

$$V = \pi \int_0^h y^2 \, dx = \pi \int_0^h [R^2 - (K_1 X)^2] \, dx$$

$$= \pi \left[R^2 h - \frac{1}{3} K_1^2 h^3 \right] \qquad K_1 = R/a$$

$$= \underline{\pi h \left(\frac{R}{a}\right)^2 \left(a^2 - \frac{1}{3} h^2\right)}$$

VOLUME ENCLOSED TO a (ENTIRE BULK'D)

$$V = \pi \left[R^2 h - \frac{1}{3} K_1^2 h^3 \right] \qquad \text{(FROM ABOVE)}$$

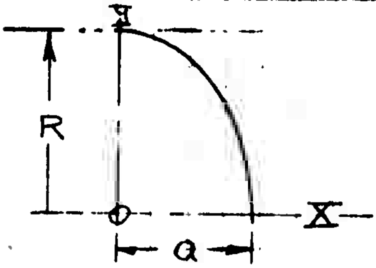
WHEN $h = a$ AND $K_1 = R/a$

$$V = \pi \left[R^2 a - \frac{1}{3} \frac{R^2}{a^2} \cdot a^3 \right]$$

$$= \underline{2/3 \, \pi R^2 a}$$

ELLIPTICAL BULKHEAD (CONT'D)

SURFACE AREA (BULK'D IN CYLINDRICAL TANK)



$$y^2 = R^2 - (K_1 x)^2$$

$$K_1 = R/a$$

$$2y\,dy = -2K_1^2 x\,dx$$

$$\frac{dy}{dx} = -\frac{K_1^2 x}{y}$$

$$S = 2\pi \int_0^a y\,ds \qquad ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

$$= 2\pi \int_0^a y \sqrt{1 + \frac{K_1^4 x^2}{y^2}}\,dx$$

$$= 2\pi \int_0^a \sqrt{y^2 + K_1^4 x^2}\,dx$$

$$= 2\pi \int_0^a \sqrt{R^2 - K_1^2 x^2 + K_1^4 x^2}\,dx$$

$$= 2\pi \int_0^a \sqrt{R^2 + (K_1^4 - K_1^2)\,x^2}\,dx$$

LET $C_1 = K_1^4 - K_1^2$

$$= 2\pi \int_0^a \sqrt{C_1 x^2 + R^2}\,dx$$

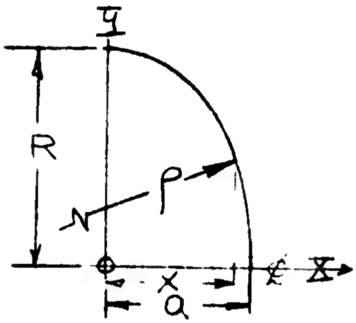
$$= 2\pi \left[\frac{x}{2}\sqrt{C_1 x^2 + R^2} + \frac{R^2}{2\sqrt{C_1}} \operatorname{LOG}\left(x\sqrt{C_1} + \sqrt{C_1 x^2 + R^2}\right)\right]_0^a$$

$$\boxed{S = \pi \left\{ a\sqrt{C_1 a^2 + R^2} + \frac{R^2}{\sqrt{C_1}}\left[\operatorname{LOG}\left(a\sqrt{C_1} + \sqrt{C_1 a^2 + R^2}\right) - \operatorname{LOG} R\right]\right\}}$$

a = BULK'D HEIGHT $\quad R$ = TANK RADIUS
$C_1 = K_1^4 - K_1^2$ WHERE $K_1 = R/a$
LOG. IS TO BASE "e"

ELLIPTICAL BULKHEAD (CONT'D)

RADIUS OF CURVATURE (ρ)



$$y^2 = R^2 - (K_1 X)^2$$

$$K_1 = R/a$$

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

$$y = (R^2 - K_1^2 X^2)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = -K_1^2 X (R^2 - K_1^2 X^2)^{-\frac{1}{2}}$$

$$\frac{d^2y}{dx^2} = \frac{(R^2-K_1^2X^2)^{1/2}(-K_1^2) - (-K_1^2X)\frac{1}{2}(R^2-K_1^2X^2)^{-\frac{1}{2}}(-2K_1^2X)}{R^2 - K_1^2 X^2}$$

$$= \frac{(R^2-K_1^2X^2)^{-\frac{1}{2}}\left[(R^2-K_1^2X^2)(-K_1^2) - (-K_1^2X)\frac{1}{2}(-2K_1^2X)\right]}{R^2 - K_1^2 X^2}$$

$$= \frac{-K_1^2(R^2-K_1^2X^2) - K_1^4X^2}{(R^2-K_1^2X^2)^{3/2}} = \frac{-K_1^2R^2}{(R^2-K_1^2X^2)^{3/2}}$$

$$\rho = \frac{\left[1 + \left\{-K_1^2X(R^2-K_1^2X^2)^{-\frac{1}{2}}\right\}^2\right]^{3/2}}{\frac{-K_1^2R^2}{(R^2-K_1^2X^2)^{3/2}}}$$

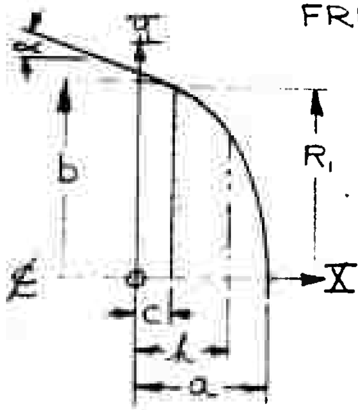
$$= \frac{(R^2-K_1^2X^2)^{3/2}\left[1 + K_1^4X^2(R^2-K_1^2X^2)^{-1}\right]^{3/2}}{-K_1^2R^2}$$

$$= -\frac{\left[R^2 + K_1^2X^2(K_1^2-1)\right]^{3/2}}{K_1^2R^2}$$

82.8
31.5

ELLIPTICAL BULKHEAD (CONT'D)

VOLUME (ENCLOSED BY THE BULKHEAD AT THE END OF A CONICAL FRUSTUM TANK)



GEN ELLIPTICAL EQUATION

$$\frac{X^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$y^2 = b^2 - (K_2 X)^2$$

$$K_2 = b/a$$

WHERE:

b = MAJOR SEMI·AXIS

a = MINOR SEMI AXIS

h = PROPELLANT LEVEL

$C = \frac{R_1 \tan\alpha}{K_2^2}$ (INTERSECTION OF TANK WITH BULK'D)

VOLUME ENCLOSED FROM "C" TO "h"

$$V = \pi\int_c^h y^2\,dx = \pi\int_c^h \left[b^2 - (K_2 X)^2\right] dx$$

$$= \pi\left[b^2 h - \tfrac{1}{3}K_2^2 h^3 - b^2 c + \tfrac{1}{3}K_2^2 c^3\right]$$

$$= \pi\left[b^2(h-c) + \tfrac{1}{3}K_2^2(c^3 - h^3)\right]$$

(OR) $$= \pi\left(\frac{b}{a}\right)^2\left[a^2(h-c) + \tfrac{1}{3}(c^3 - h^3)\right] \qquad K_2 = b/a$$

VOLUME ENCLOSED FROM "C" TO "a"

$$V = \pi\left[b^2(a-c) + \tfrac{1}{3}K_2^2(c^3 - a^3)\right] \qquad K_2 = b/a$$

(OR) $$= \pi\left(\frac{b}{a}\right)^2\left[a^2(a-c) + \tfrac{1}{3}(c^3 - a^3)\right]$$

ELLIPTICAL BULKHEAD (CONT'D)

(BULK'D AT END OF CONICAL FRUSTUM TANK CONT'D)

GENERALLY SPEAKING THE VALUE K_2 (RATIO OF MAJOR TO MINOR AXIS) IS PRE-DETERMINED BY STRUCTURAL CONSIDERATIONS. ALSO THE VALUE OF R_1 (RADIUS OF TANK END) AND α (CONICAL FRUSTUM HALF ANGLE) ARE KNOWN FROM THE TANK GEOMETRY. FROM THESE (K_2, R_1 & α) THE BULKHEAD GEOMETRY MAY BE SET AS FOLLOWS.

BULKHEAD'S MAJOR AXIS "b"

$$y^2 = b^2 - (K_2 X)^2 \qquad \text{(BASIC EQUATION)}$$

$$b = \sqrt{y^2 + (K_2 X)^2}$$

WHEN $y = R_1 \quad X = C = \dfrac{R_1 \tan\alpha}{K_2^2}$

$$b = \sqrt{R_1^2 + \left(K_2 \frac{R_1 \tan\alpha}{K_2^2}\right)^2}$$

$$= \frac{R_1}{K_2}\sqrt{K_2^2 + \tan^2\alpha}$$

$$a = b/K_2$$

IF THE BULK'DS HEIGHT "H" BEYOND THE TANK END IS KNOWN IN ADDITION TO K_2 R_1 & α THEN,

$$H = a - C \qquad \text{(FROM FIGURE OF PREVIOUS PAGE)}$$

$$= a - \frac{R_1 \tan\alpha}{K_2^2}$$

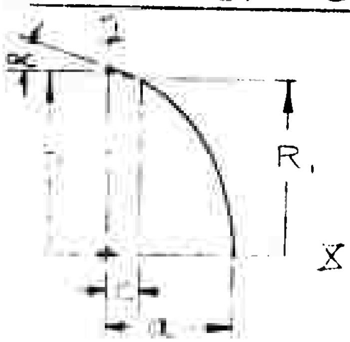
$$a = H + \frac{R_1 \tan\alpha}{K_2^2}$$

$$b = K_2 a$$

ELLIPTICAL BULKHEAD (CONT'D)

(BULK'D AT END OF CONICAL FRUSTUM TANK-CONT'D)

VALUE OF "C" AND "R_1"



C = DISTANCE BETWEEN MAJOR AXIS AND BULKHEAD INTERSECTION WITH TANK

R_1 = TANK AND BULKHEAD INTERSECTION RADIUS (RADIUS AT TOP OF TANK)

a = BULK'D MINOR SEMI AXIS

b = " MAJOR " "

$$y^2 = b^2 - (K_2 x)^2 \quad \text{BULK'D GENERAL FORMULA}$$

$$K_2 = b/a$$

$\frac{dy}{dx}$ = SLOPE AT ANY POINT ON ELLIPSE

$$2y\,dy = 0 - 2K_2^2 x\,dx \quad \text{(DIFF. OF ABOVE FORMULA)}$$

$$\frac{dy}{dx} = -\frac{2K_2^2 x}{2y} = -K_2^2 \frac{x}{y}$$

ALSO $\frac{dy}{dx} = -\text{TAN}\,\alpha$

THEN $$-\text{TAN}\,\alpha = -K_2^2 \frac{x}{y}$$

WHEN $x = C \quad y = R_1$

$$\text{TAN}\,\alpha = K_2^2 \frac{C}{R_1}$$

$$C = \frac{R_1 \,\text{TAN}\,\alpha}{K_2^2}$$

ELLIPTICAL BULKHEAD (CONT'D)

(BULK'D AT END OF CONICAL FRUSTUM TANK - CONT'D)

VALUE OF "C" AND "R_1" (CONT'D)

$$\tan\alpha = K_2^2 \frac{x}{y} \qquad \text{(FROM PREVIOUS PAGE)}$$

WHEN $x = C$ $\quad y^2 = b^2 - (K_2 C)^2$

$$\tan^2\alpha = K_2^4 \frac{x^2}{y^2}$$

$$\tan^2\alpha\,(b^2 - K_2^2 C^2) = K_2^4 C^2$$

$$(K_2^4 + K_2^2 \tan^2\alpha)\,C^2 = b^2 \tan^2\alpha$$

$$C = \frac{b \tan\alpha}{K_2\sqrt{K_2^2 + \tan^2\alpha}} = \frac{b \tan\alpha}{\frac{b}{a}\sqrt{\frac{b^2}{a^2} + \tan^2\alpha}}$$

$$= \frac{a^2 \tan\alpha}{\sqrt{b^2 + a^2 \tan^2\alpha}}$$

WHEN $x = C$ $\quad y = R_1$

THEN FROM $y^2 = b^2 - K_2^2 x^2$

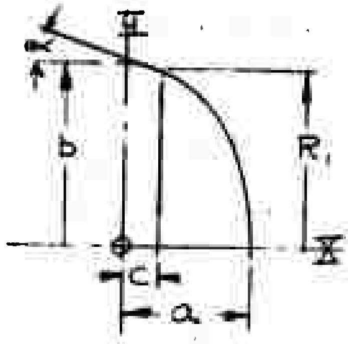
$$R_1^2 = b^2 - K_2^2 C^2$$

$$R_1 = \sqrt{b^2 - \frac{b^2}{a^2} C^2}$$

$$= \frac{b}{a}\sqrt{a^2 - C^2}$$

ELLIPTICAL BULKHEAD (CONT'D)

SURFACE AREA (BULK'D AT SMALL END OF CONICAL FRUSTUM TANK)



$$y^2 = b^2 - (K_2 x)^2$$

$$K_2 = b/a$$

SURFACE AREA OF THE ELLIPTICAL BULK'D FROM LINE OF TANK INTERSECTION IS

$$S = \int_c^a y\, ds$$

FROM WHICH, (SEE DEVEL. ON PREVIOUS PAGE)

$$S = 2\pi \int_c^a \sqrt{C_2 x^2 + b^2}\; dx$$

WHERE $C_2 = K_2^4 - K_2^2$

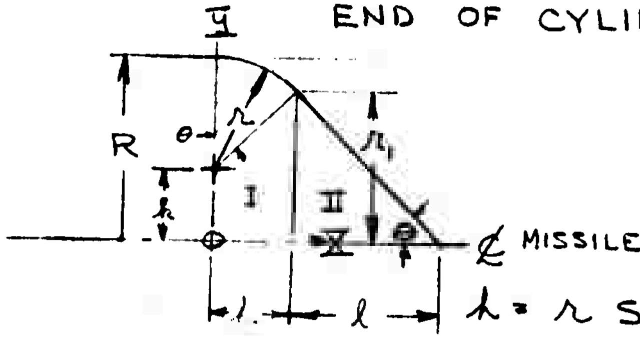
$$= 2\pi \left[\frac{x}{2}\sqrt{C_2 x^2 + b^2} + \frac{b^2}{2\sqrt{C_2}} \operatorname{LOG}\left(x\sqrt{C_2} + \sqrt{C_2 x^4 + b^2} \right) \right]_c^a$$

$$\boxed{\begin{aligned} S = \pi \Big[& a\sqrt{C_1 a^2 + b^2} + \frac{b^2}{\sqrt{C_2}} \operatorname{LOG}\left(a\sqrt{C_2} + \sqrt{C_2 a^2 + b^2}\right) \\ & - c\sqrt{C_1 c^2 + b^2} - \frac{b^2}{\sqrt{C_2}} \operatorname{LOG}\left(c\sqrt{C_2} + \sqrt{C_2 c^2 + b^2}\right) \Big] \end{aligned}}$$

WHERE: a = BULK'D MINOR SEMI AXIS (HEIGHT)
b = " MAJOR " "
C = DIST. BETWEEN MAJOR AXIS AND BULK'D INTERSECT WITH TANK = $\frac{R_1 \tan\alpha}{K_2^2}$
$C_1 = K_2^4 - K_2^2$ WHERE $K_2 = b/a$
LOG IS TO BASE "e"

TORICONICAL BULKHEAD

VOLUME (ENCLOSED BY BULKHEAD AT END OF CYLINDRICAL TANK)



$h = r \sin\theta$

VOLUME I

$$x^2 + (y-k)^2 = r^2 \qquad y = k + \sqrt{r^2 - x^2}$$

$$y^2 = k^2 + 2k\sqrt{r^2-x^2} + r^2 - x^2$$

$$V_I = \pi\int_0^l y^2\,dx = \pi\int_0^l \left(k^2 + 2k\sqrt{r^2-x^2} + r^2 - x^2\right)dx$$

$$= \pi\left\{k^2 x + 2k\,\tfrac{1}{2}\left[x\sqrt{r^2-x^2} + r^2\sin^{-1}\left(\frac{x}{r}\right)\right] + r^2 x - \frac{x^3}{3}\right\}_0^l$$

$$= \underline{\underline{\pi\left\{l(r^2+k^2) + k\left[l\sqrt{r^2-l^2} + r^2\sin^{-1}\left(\frac{l}{r}\right)\right] - \frac{l^3}{3}\right\}}}$$

VOLUME II

$V = \frac{1}{3} A_B\, l$ (GENERAL CONE FORMULA)

$= \frac{1}{3}\pi r_1^2 \times r_1 \cot\theta$

$= \underline{\underline{\frac{1}{3}\pi r_1^3 \cot\theta}}$

$r_1 = R - r + h\cot\theta$

$= k + h\cot\theta$

OR $= k + r\cos\theta$

TORICONICAL BULK'D (CONT'D)

SURFACE AREA

SURFACE ENCLOSING VOLUME I

$$S_I = 2\pi\int_0^{l} y\,ds \qquad ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$y = -h + \sqrt{r^2 - x^2}$$

$$\frac{dy}{dx} = \frac{x}{\sqrt{r^2 - x^2}} \qquad \left(\frac{dy}{dx}\right)^2 = \frac{x^2}{r^2 - x^2}$$

$$S_I = 2\pi\int_0^{l}\left(h + \sqrt{r^2 - x^2}\right)\left[1 + \frac{x^2}{r^2 - x^2}\right]^{\frac{1}{2}} dx$$

$$= 2\pi\int_0^{l}\left(h + \sqrt{r^2 - x^2}\right)\left(\frac{r}{\sqrt{r^2 - x^2}}\right) dx$$

$$= 2\pi\int_0^{l}\left(\frac{hr}{\sqrt{r^2 - x^2}} + \sqrt{r^2 - x^2} \times \frac{r}{\sqrt{r^2 - x^2}}\right) dx$$

$$= 2\pi r\int_0^{l}\left[\frac{h}{\sqrt{r^2 - x^2}} + 1\right] dx$$

$$= 2\pi r\left[h \sin^{-1}\left(\frac{x}{r}\right) + x\right]_0^{l}$$

$$= 2\pi r\left[h \sin^{-1}\left(\frac{l}{r}\right) + l\right]$$

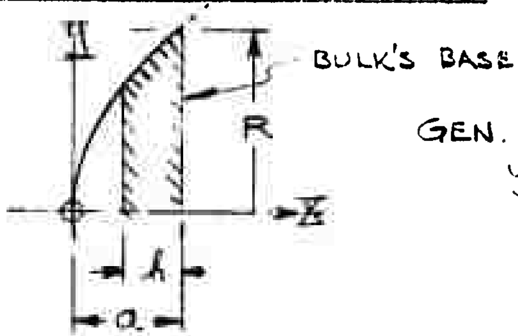
SURFACE ENCLOSING VOLUME II

$$S_{II} = \frac{\text{AREA OF BASE}}{\sin\theta} = \frac{\pi r_1^2}{\sin\theta} \qquad \text{(CLASSIC)}$$

TOTAL SURF. AREA OF TORICONICAL BULK'D IS;

$$S_T = S_I + S_{II}$$

PARABOLIC BULKHEAD



GEN. PARABOLIC EQUATION

$$y^2 = AX$$

$$A = \frac{R^2}{a}$$

VOLUME ENCLOSED FROM "h" TO "a" (SHADED AREA)

$$V = \pi\int_{a-h}^{h} y^2\,dx = \pi\int_{a-h}^{h} AX\,dx$$

$$= \pi\frac{R^2}{a}\int_{a-h}^{h} X\,dX$$

$$= \frac{\pi R^2}{2a}X^2\Big]_{a-h}^{a} = \frac{\pi R^2}{2a}\left[a^2-(a-h)^2\right]$$

$$= \tfrac{1}{2}\pi R^2\left[2h-\frac{h^2}{a}\right] = \underline{\underline{\tfrac{1}{2}\pi R^2 h\left(2-\frac{h}{a}\right)}}$$

WHEN $h = a$ THEN,

$$V_T = \underline{\underline{\tfrac{1}{2}\pi R^2 a}}$$

SURFACE AREA OF ENTIRE BULK'D

$$S = 2\pi\int_0^a y\,ds \qquad ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= 2\pi\int_0^a y\sqrt{1+\left(\frac{A}{2y}\right)^2}\,dx = \pi\int_0^a\sqrt{4AX+A^2}\,dx$$

$$= \pi\left[\frac{2}{3\times 4A}(4AX+A^2)^{3/2}\right]_0^a = \frac{\pi}{6A}\left[(4Aa+A^2)^{3/2}-A^3\right]$$

$$= \frac{\pi R^3}{6A}\left[\left(4+\frac{R^2}{a^2}\right)^{3/2}-\frac{R^3}{a^3}\right] = \underline{\frac{\pi Ra}{6}\left[\left(4+\frac{R^2}{a^2}\right)^{3/2}-\left(\frac{R}{a}\right)^3\right]}$$

PARABOLIC BULK'D CONT'D

RADIUS OF CURVATURE (ρ)

$$y^2 = Ax \qquad A = R^2/a$$

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

$$y = A^{\frac{1}{2}}x^{\frac{1}{2}} \qquad \frac{dy}{dx} = \frac{1}{2}A^{\frac{1}{2}}x^{-\frac{1}{2}}$$

$$\frac{d^2y}{dx^2} = -\frac{1}{4}A^{\frac{1}{2}}x^{-\frac{3}{2}}$$

$$\rho = \frac{\left[1 + \left(\frac{1}{2}A^{\frac{1}{2}}x^{-\frac{1}{2}}\right)^2\right]^{3/2}}{-\frac{1}{4}A^{\frac{1}{2}}x^{-3/2}}$$

$$= \frac{\left[1 + \frac{1}{4}Ax^{-1}\right]^{3/2}}{-\frac{1}{4}A^{\frac{1}{2}}x^{-3/2}}$$

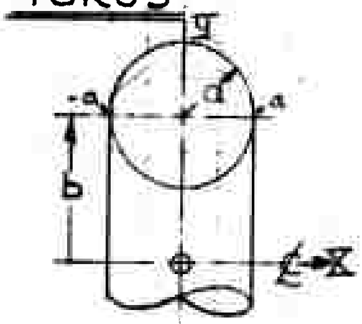
$$= \frac{-4x^{\cancel{3/2}}\left(\frac{4x + A}{4x}\right)^{3/2}}{A^{1/2}} = \frac{\overset{*}{-}4\cancel{x^{3/2}}(4x + A)^{3/2}}{8\cancel{x^{3/2}}\,A^{\frac{1}{2}}}$$

$$A = \frac{R^2}{a} \qquad = \frac{1}{2}\,\frac{\left(4x + \frac{R^2}{a}\right)^{3/2}}{\left(\frac{R^2}{a}\right)^{\frac{1}{2}}}$$

$$= \frac{1}{2Ra}\left(4ax + R^2\right)^{3/2}$$

* NOTE: THE MINUS SIGN APPEARING HAS NO SIGNIFIGANCE AND IS THEFORE DROPPED FROM THE EQUATION.

TORUS



$$x^2 + (y-b)^2 = a^2$$

$$y - b = \pm\sqrt{a^2 - x^2}$$

$$y = b \pm \sqrt{a^2 - x^2}$$

$$\frac{dy}{dx} = \mp x(a^2 - x^2)^{-\frac{1}{2}}$$

$y_u = b + \sqrt{a^2 - x^2}$ IS THE EQUATION FOR THE UPPER SEMICIRCLE
$y_L = b - \sqrt{a^2 - x^2}$ " " " " " LOWER "

SURFACE AREA

$$S = 2\pi \int y \, ds \qquad ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

AREA GENERATED BY REVOLVING THE UPPER SEMI-CIRCLE

$$S_u = 2\pi \int_{-a}^{a} \left(b + \sqrt{a^2 - x^2}\right)\left[1 + \frac{x^2}{a^2 - x^2}\right]^{1/2} dx$$

AREA GENERATED BY REVOLVING THE LOWER SEMI-CIRCLE

$$S_L = 2\pi \int_{-a}^{a} \left(b - \sqrt{a^2 - x^2}\right)\left[1 + \frac{x^2}{a^2 - x^2}\right]^{1/2} dx$$

TOTAL SURFACE AREA $S_T = S_U + S_L$

$$S_T = 2\pi \int_{-a}^{a} y_u \, ds + 2\pi \int_{-a}^{a} y_L \, ds = 2\pi \int_{-a}^{a} (y_u + y_L) \, ds$$

$$= 2\pi \int_{-a}^{a} \left[\left(b + \cancel{\sqrt{a^2 - x^2}}\right) + \left(b - \cancel{\sqrt{a^2 - x^2}}\right)\right] ds$$

$$= 2\pi \int_{-a}^{a} 2b \, ds$$

$$= 4\pi b \int_{-a}^{a} \left[1 + \frac{x^2}{a^2 - x^2}\right]^{1/2} dx = 4\pi b \int_{-a}^{a} \left(\frac{a^2}{a^2 - x^2}\right)^{\frac{1}{2}} dx$$

$$= 4\pi a b \int_{-a}^{a} (a^2 - x^2)^{-1/2} dx$$

TORUS (CONT'D)

$$S_T = 4\pi a b \left(\sin^{-1}\frac{x}{a}\right) + C \Big]_{-a}^{a}$$

$$= 4\pi a b \left[\sin^{-1}(1) - \sin^{-1}(-1)\right]$$

$$= 4\pi a b \left[2 \cdot \frac{\pi}{2}\right]$$

$$= \underline{\underline{4\pi^2 a b}}$$

VOLUME (ENCLOSED BY TORUS)

$$V_T = \pi \int_{-a}^{a} (y_U^2 - y_L^2)\, dx$$

$$= \pi \int_{-a}^{a} \left[(b + \sqrt{a^2 - x^2})^2 - (b - \sqrt{a^2 - x^2})^2\right] dx$$

$$= \pi \int_{-a}^{a} \left[\cancel{b^2} + 2b\sqrt{\quad} + \cancel{a^2} \cancel{- x^2} \cancel{- b^2} + 2b\sqrt{\quad} \cancel{- a^2} \cancel{+ x^2}\right] dx$$

$$= 4\pi b \int_{-a}^{a} \sqrt{a^2 - x^2}\, dx$$

$$= 4\pi b \cdot \tfrac{1}{2}\left[x\sqrt{a^2 - x^2} + a^2 \sin^{-1}\left(\frac{x}{a}\right)\right]_{-a}^{a}$$

$$= 2\pi b \left[a^2 \sin^{-1}(1) - a^2 \sin^{-1}(-1)\right]$$

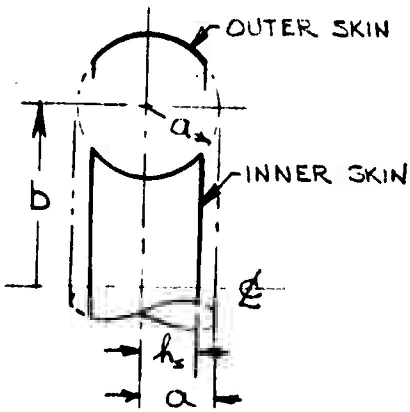
$$= 2\pi b \left[a^2 \times \frac{\pi}{2} - a^2 \times \left(-\frac{\pi}{2}\right)\right]$$

$$= 2\pi b \left[2a^2 \times \frac{\pi}{2}\right]$$

$$= \underline{\underline{2\pi^2 b a^2}}$$

TORUS (CONT'D)

THE DESIGN OF A TORUS TANK IS USUALLY SUCH THAT HIGH STRESSES ARE DEVELOPED IN THE SKIN OF THE "DO-NUT" HOLE WHILE THE TANK IS UNDER PRESSURE. THIS OFTEN DICTATES HEAVIER GAGE SKINS IN THIS AREA. ALSO LIGHTER SKINS ON THE OUTSIDE OF THE TORUS. THE FOLLOWING FORMULAS ARE DEVELOPED FOR THESE LOCAL AREAS.



$$x^2 + (y-b)^2 = a^2$$

$$\begin{cases} y_o = b + \sqrt{a^2 - x^2} & \text{(UPPER)} \\ y_i = b - \sqrt{a^2 - x^2} & \text{(LOWER)} \end{cases}$$

SURFACE AREA OF OUTER SKIN ($2\,h_s$ WIDE)

$$S_o = 2 \times 2\pi \int_0^{h_s} y_o \, ds \qquad ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= 4\pi \int_0^{h_s} \left(b + \sqrt{a^2 - x^2}\right) \left[1 + \frac{x^2}{a^2 - x^2}\right]^{1/2} dx$$

$$= 4\pi \int_0^{h_s} \left(b + \sqrt{a^2 - x^2}\right) \left[\frac{a}{\sqrt{a^2 - x^2}}\right] dx$$

$$= 4\pi \int_0^{h_s} \left(\frac{ab}{\sqrt{a^2 - x^2}} + \frac{a\sqrt{a^2 - x^2}}{\sqrt{a^2 - x^2}}\right) dx$$

$$= 4\pi a \left[b \sin^{-1}\left(\frac{x}{a}\right) + x \right]_0^{h_s}$$

$$= \underline{4\pi a \left(b \sin^{-1}\left(\frac{h_s}{a}\right) + h_s \right)} \quad *$$

TORUS (CONT'D)

SURFACE AREA OF INNER SKIN ($2h_s$ WIDE)

$$S_i = 2 \times 2\pi \int_0^{h_s} y_i \, ds$$

$$= 4\pi \int_0^{h_s} \left(b - \sqrt{a^2 - x^2}\right)\left[1 + \frac{x^2}{a^2 - x^2}\right]^{1/2} dx$$

$$= 4\pi a \int_0^{h_s} \left(\frac{b}{\sqrt{a^2 - x^2}} - 1\right) dx$$

$$= 4\pi a \left[b \sin^{-1}\left(\frac{x}{a}\right) - x\right]_0^{h_s}$$

$$= \underline{\underline{4\pi a \left(b \sin^{-1}\left(\frac{h_s}{a}\right) - h_s\right)}} \quad *$$

CHECK FOR ABOVE S_o' & S_i' EQUATIONS:

$$S_T = S_o + S_i \qquad \text{WHEN } h_s = a$$

$$= 4\pi a \left(b \sin^{-1}\left(\frac{h_s}{a}\right) + h\right) + 4\pi a \left(b \sin^{-1}\frac{h_s}{a} - h_s\right)$$

$$= 4\pi a \left(b \sin^{-1}(1) \cancel{+a} + b \sin^{-1}(1) \cancel{-a}\right)$$

$$= 4\pi a \left(2b \times \frac{\pi}{2}\right)$$

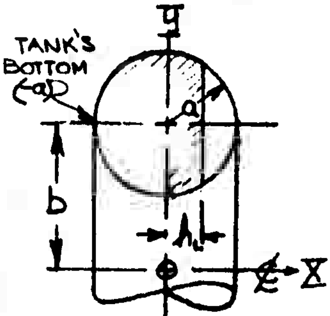
$$= 4\pi^2 a b$$

(THIS CHECKS WITH FORMULA FOR TOTAL AREA OF TORUS)

* NOTE: $\sin^{-1}\left(\frac{h_s}{a}\right)$ IS DEFINED AS THE ANGLE WHOSE SINE IS h_s/a. TO EVALUATE THIS FUNCTION FIND h_s/a VALUE IN A TRIG. SINE TABLE TO OBTAIN IT'S CORRESPONDING ANGLE IN DEGREES. MULTIPLY THE DEGREES BY .01745 TO OBTAIN RADIANS WHICH IS PLUGGED DIRECTLY INTO THE FORMULA IN PLACE OF "$\sin^{-1}\left(\frac{h_s}{a}\right)$".

TORUS (CONT'D)

VOLUME OF PARTIALLY FILLED TORUS TANK



$$x^2 + (y-b)^2 = a^2$$

VOLUME OF FULL TANK

$$= 2\pi^2 b a^2$$ (SEE PG. 23)

$$V' = \pi \int_{-a}^{h} (y_u^2 - y_l^2)\, dx$$ (SEE PG. 23)

$$= 4\pi b \cdot \tfrac{1}{2}\left[x\sqrt{a^2 - x^2} + a^2 \sin^{-1}\left(\frac{x}{a}\right)\right]_{-a}^{h_L}$$

$$= 2\pi b \left[h_L\sqrt{a^2 - h_L^2} + a^2 \sin^{-1}\left(\frac{h_L}{a}\right) - a^2 \sin^{-1}(-1)\right]$$

$$= 2\pi b \left\{ h_L\sqrt{a^2 - h_L^2} + a^2\left[\sin^{-1}\left(\frac{h_L}{a}\right) + \frac{\pi}{2}\right]\right\}$$

"h_L" MAY BE (+) OR (−)

IF "h_L" IS REGARDED AS A PROPORTION OF "a" THEN $h_L = Ka$. SUBSTITUTING THIS IN THE FORMULA ABOVE, THEN:

$$V_L = 2\pi a^2 b \left\{ K\sqrt{1 - K^2} + \sin^{-1} K + \frac{\pi}{2} \right\}$$

$$K = h_L / a$$